Integrable particle systems and Macdonald processes

Ivan Corwin

(Columbia University, Clay Mathematics Institute, Massachusetts Institute of Technology and Microsoft Research)

Lecture 2

- Introduction to Schur polynomials
- Definition of Schur measure and process
- Dynamics which preserve class of Schur measure / process
- Connections to TASEP and LPP
- Schur measure determinantal point process kernel
- Limit theorem for TASEP

Partitions

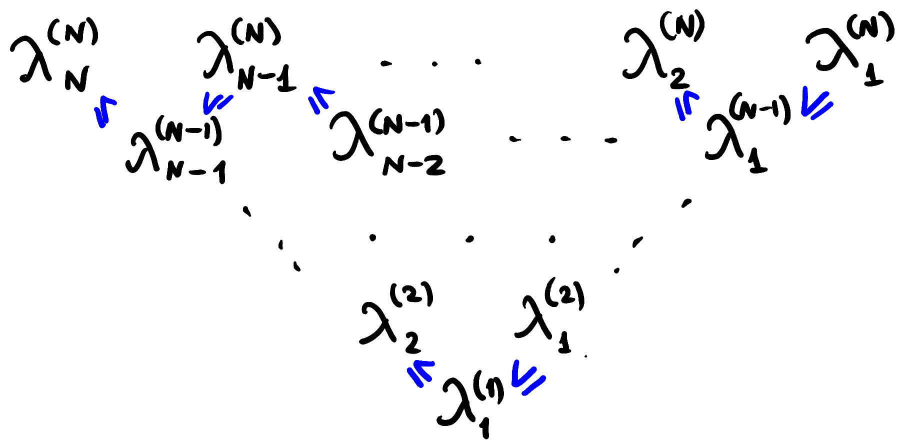
Partition: $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ weakly decreasing with $\lambda_i \in \mathbb{Z}_{\geq 0}$

length $\ell(\lambda) = |\{i : \lambda_i > 0\}|$ and size $|\lambda| = \sum_i \lambda_i$.

(e.g. $\lambda = (4, 2, 1, 1)$, $\ell(\lambda) = 4$, $|\lambda| = 8$)

Interlacing: $\lambda \succeq \mu$ if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all $i \geq 1$.

Gelfand-Tsetlin schemes: $\lambda^{(N)} \succeq \lambda^{(N-1)} \succeq \cdots \succeq \lambda^{(1)}$ with $\ell(\lambda^{(i)}) \leq i$.



(e.g. $\begin{matrix} 1 & & 2 & & 4 \\ & 2 & & 3 & \\ & & 2 & & \end{matrix} \longleftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$)

Ex: Show GT schemes are same as semi-standard Young tableaux.

Schur polynomials

Schur symmetric polynomial (Issai Schur, 1900)

$$s_\lambda(x_1,\ldots,x_N) := \frac{\det\left[x_i^{N+\lambda_j-j}\right]_{i,j=1}^{N}}{\det\left[x_i^{N-j}\right]_{i,j=1}^{N}}$$

Vandermonde determinant

Ex: Prove that these are symmetric polynomials. Compute $s_{(3,1)}(x_1,x_2,x_3)$.

Multivariate symmetric polynomials which form linear basis of space of symmetric polynomials. Important role in representation theory. Have many nice properties (some of which we will use).

Branching rule

$$S_\lambda(x_1,\dots,x_N) = \sum_{\mu \preceq \lambda} S_\mu(x_1,\dots,x_{N-1})\; x_N^{|\lambda|-|\mu|}$$

$$S_{\lambda/\mu}(x_N) := x_N^{|\lambda|-|\mu|}\,\mathbb{1}_{\lambda \succeq \mu}$$

Iterating the branching rule gives the combinatorial formula

$$S_\lambda(x_1,\dots,x_N) = \sum_{\lambda^{(1)} \preceq \cdots \preceq \lambda^{(N-1)} \preceq \lambda^{(N)} = \lambda} x_N^{|\lambda^{(N)}|-|\lambda^{(N-1)}|}\, x_{N-1}^{|\lambda^{(N-1)}|-|\lambda^{(N-2)}|} \cdots x_1^{|\lambda^{(1)}|}$$

All GT-schemes with top line λ

Thus, for $x_1,\dots,x_N \geq 0$ we have $S_\lambda(x_1,\dots,x_N) \geq 0$ (positivity)

Ex: Prove branching rule. Compute the number of GT-schemes with top row λ.
Use this to rederive yesterday's result on the volume of interlacing triangular arrays.

Schur measure [Okounkov, 2001]

A probability measure on partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$ given by

$$\mathbb{SM}_{X;Y}(\lambda) = \frac{s_\lambda(X)\, s_\lambda(Y)}{\Pi(X;Y)}$$

where $X = \{x_1, \ldots, x_N\}$ and $Y = \{y_1, \ldots, y_M\}$ are positive parameters.

Cauchy-Littlewood identity evaluates partition function as

$$\Pi(X;Y) = \sum_\lambda s_\lambda(X)\, s_\lambda(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

Ex: Prove above identity using the Cauchy determinant identity.

Discrete (X,Y)-parameter generalization of GUE eigenvalue measure

Schur process [Okounkov-Reshetikhin, 2001]

A probability measure on GT-schemes $\lambda^{(N)} \succ \cdots \succ \lambda^{(1)}$ given by

$$\mathbb{S}_{X;Y}\left(\lambda^{(N)}, \ldots, \lambda^{(1)}\right) = \frac{s_{\lambda^{(N)}}(Y)\, s_{\lambda^{(N)}/\lambda^{(N-1)}}(x_N)\, s_{\lambda^{(N-1)}/\lambda^{(N-2)}}(x_{N-1}) \cdots s_{\lambda^{(1)}}(x_1)}{\Pi(X;Y)}$$

$$\begin{array}{ccccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & \lambda^{(N)}_1 \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & \cdots & & & \lambda^{(N-1)}_1 & \\
& & \ddots & & \cdots & & \iddots & & \\
& & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & \\
& & & & \lambda^{(1)}_1 & & & &
\end{array}$$

$$s_{\lambda/\mu}(u) := u^{|\lambda|-|\mu|}\, \mathbb{1}_{\lambda \succ \mu}$$

Fact: Level k marginal distributed as $\mathbb{SM}_{\{x_1, \ldots, x_k\};Y}$.

Discrete (X,Y)-parameter generalization of GUE corner process

Gibbs property

If all $x_i = 1$ then levels $N-1, \dots, 1$ are marginally distributed uniformly over GT-schemes with top level $\lambda^{(N)}$.

More generally, define stochastic links $\Lambda^k_{k-1}(\lambda, \mu)$, $1 \le k \le N$

$$\Lambda^k_{k-1}(\lambda, \mu) := \frac{s_\mu(x_1, \dots, x_{k-1})}{s_\lambda(x_1, \dots, x_{k-1}, x_k)} \, s_{\lambda/\mu}(x_k).$$

Schur process is distributed as the trajectory a Markov chain with these transition matrices, initially distributed as Schur measure

$$\mathbb{S}_{X;Y}(\lambda^{(N)}, \dots, \lambda^{(1)}) = \mathbb{SM}_{X;Y}(\lambda^{(N)}) \Lambda^N_{N-1}(\lambda^{(N)}, \lambda^{(N-1)}) \cdots \Lambda^1_0(\lambda^{(1)}, \varnothing)$$

Discrete time/space DBM type dynamics

Markov chain on level N which preserves class of Schur measure:

$$u \geq 0 \qquad P_u^{(N)}(\lambda^{(N)}, \mu^{(N)}) := \frac{s_{\mu^{(N)}}(X)}{s_{\lambda^{(N)}}(X)} \cdot \frac{s_{\mu^{(N)}/\lambda^{(N)}}(u)}{\Pi(u;X)}$$

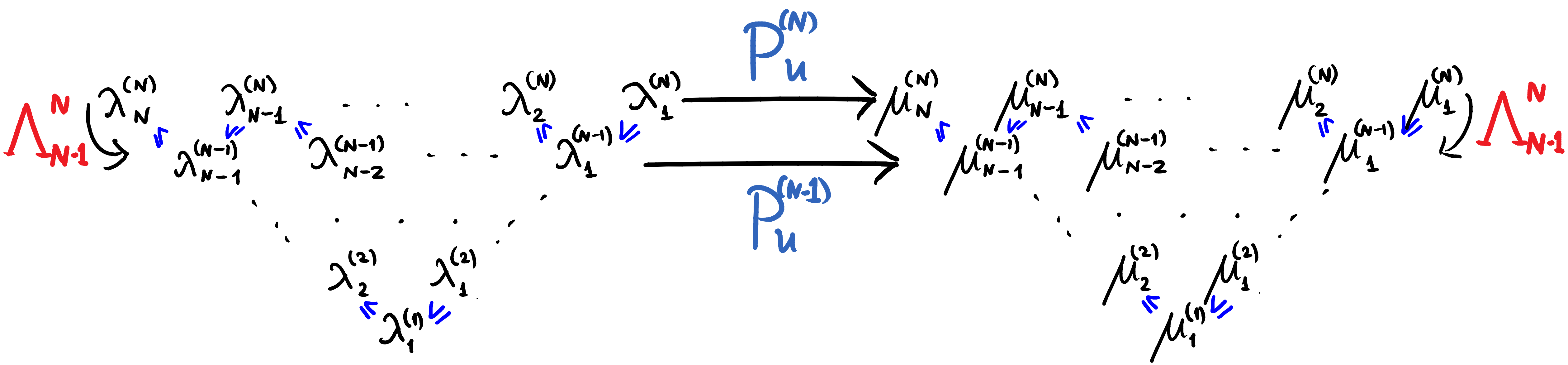
$\frac{s_{\mu^{(N)}/\lambda^{(N)}}(u)}{\Pi(u;X)}$ -> Geometric random walks killed outside Weyl chamber

$\frac{s_{\mu^{(N)}}(X)}{s_{\lambda^{(N)}}(X)}$ -> Conditioned to survive (via Doob h-transform)

Fact: The push-forward of $\mathbb{SM}_{X;Y}$ under $P_u^{(N)}$ is $\mathbb{SM}_{X;Y,u}$

Intertwining Markov dynamics

$$
\begin{tikzcd}
\mathbb{SM}_{\{x_1,\dots,x_N\};Y} \sim & \lambda^{(N)} \arrow[r, "P_u^{(N)}"] \arrow[d, "\Lambda^N_{N-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mu^{(N)} \arrow[d, "\Lambda^N_{N-1}"] & \sim \mathbb{SM}_{\{x_1,\dots,x_N\};Y,u} \\
\mathbb{SM}_{\{x_1,\dots,x_{N-1}\};Y} \sim & \lambda^{(N-1)} \arrow[r, "P_u^{(N-1)}"] & \mu^{(N-1)} & \sim \mathbb{SM}_{\{x_1,\dots,x_{N-1}\};Y,u}
\end{tikzcd}
$$



Building multivariate Markov dynamics

$$
\begin{tikzcd}
\lambda^{(N)} \arrow[r, "P_u^{(N)}"] \arrow[d, "\Lambda^N_{N-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mu^{(N)} \arrow[d, "\Lambda^N_{N-1}"] \\
\lambda^{(N-1)} \arrow[r, "P_u^{(N-1)}"] \arrow[d, dotted] & \mu^{(N-1)} \arrow[d, dotted] \\
\lambda^{(2)} \arrow[r, "P_u^{(2)}"] \arrow[d, "\Lambda^2_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mu^{(2)} \arrow[d, "\Lambda^2_1"] \\
\lambda^{(1)} \arrow[r, "P_u^{(1)}"] & \mu^{(1)}
\end{tikzcd}
$$

Due to [Diaconis-Fill, 1990, Borodin-Ferrari, 2008]

Sequentially update from bottom to top via

$$P_u(\lambda,\mu) := P_u^{(1)}(\lambda^{(1)},\mu^{(1)}) \prod_{k=2}^{N} \frac{P_u^{(k)}(\lambda^{(k)},\mu^{(k)})\,\Lambda^{k}_{k-1}(\mu^{(k)},\mu^{(k-1)})}{\left(P_u^{(k)}\Lambda^{k}_{k-1}\right)(\lambda^{(k)},\mu^{(k-1)})}$$

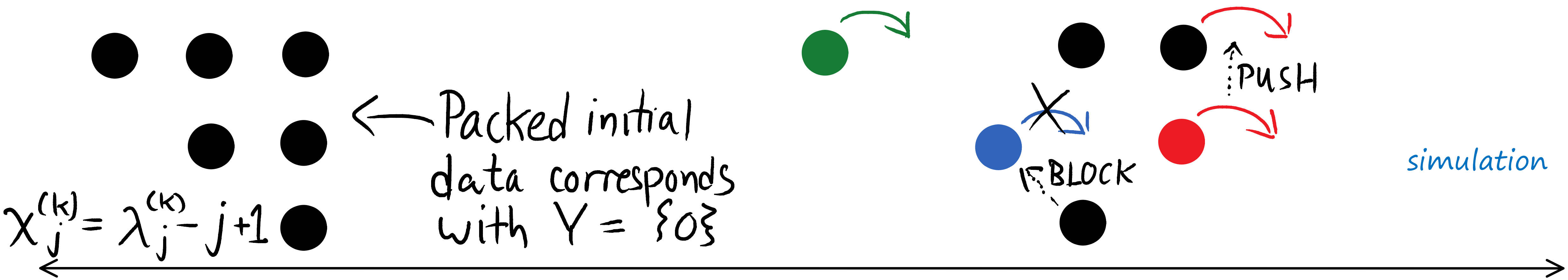
Markov chain preserves class of Schur processes

$S_{X;Y}(\lambda)$ pushes-forward via P_u to $S_{X;Y,u}(\mu)$.

Ex: Prove this fact.

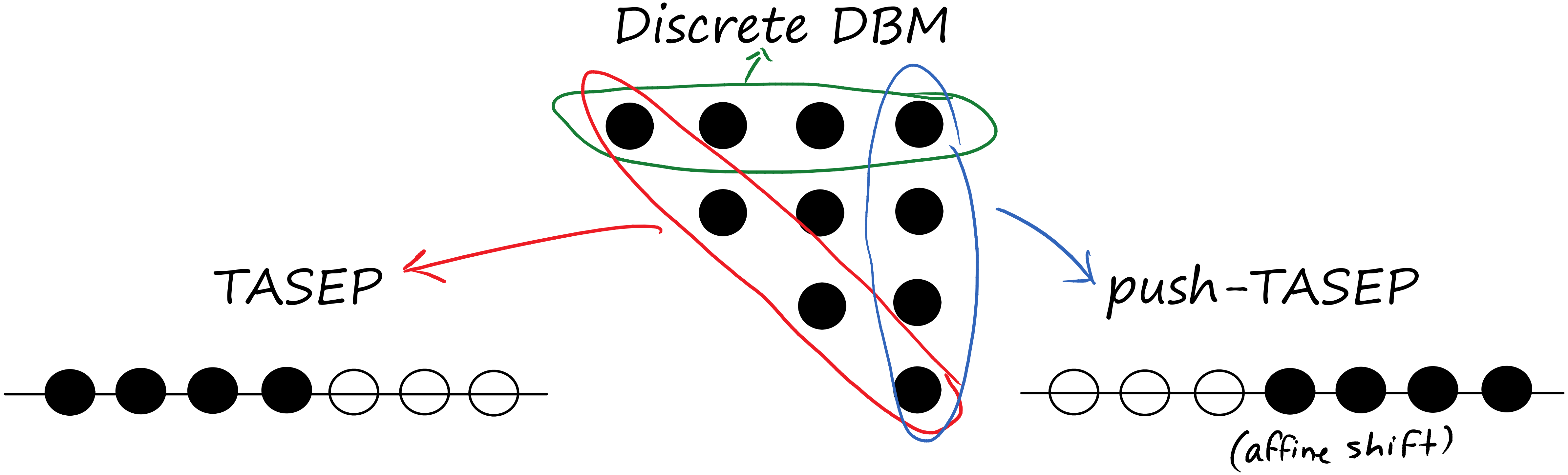
Block-push (2+1)d dynamic [Borodin-Ferrari, 2008]

Here is a continuous time dynamic corresponding to $X=\{1,\dots,1\}$ and the limit $\varepsilon\searrow 0$, $u=\varepsilon$ and taking $\varepsilon^{-1}t$ steps of the chain.
GT-scheme Schur process distributed as $\mathbb{S}_{X;\mathrm{Plancherel}(t)}$ $\lim_{\varepsilon\searrow 0}\underbrace{\{\varepsilon,\dots,\varepsilon\}}_{\varepsilon^{-1}t\text{ times}}$



Each particle jumps right at rate 1. Particles are blocked by those on the lower level, and push those on the higher level.

(1+1)d marginals

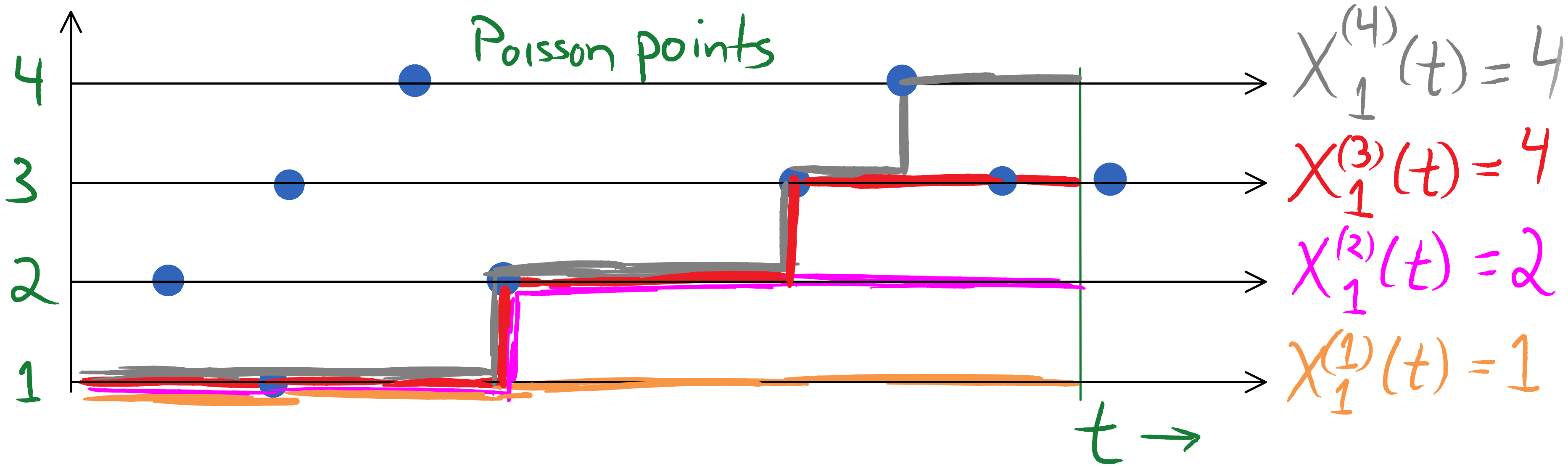


Initial data (e.g. step) corresponds to marginals of Schur processes

A further limit (taking time large and rescaling diffusively) leads to Warren's dynamics and the GUE corner process.

(2+1)d RS(K) dynamics

Ex: Prove that the right-edge (push-TASEP) marginal matches the following process in t:



This is part of the RS(K) correspondence which involves maximizing over multiple non-intersecting paths. Under RS(K) above last passage percolation model leads same Schur process.

BUT: as time changes, the (2+1)d RS(K) dynamics are different!

Determinantal point processes

Both Schur measure and process have the structure of determinantal point processes with explicit correlation kernels.

A point process $X \subseteq \{0,1\}^{\mathbb{Z}}$ is determinantal if for all k, and $x_1, \dots, x_k$

k-point Correlation function

$$\rho_k(x_1, \dots, x_k) := \mathbb{P}(\{x_1, \dots, x_k\} \subseteq X) = \det\left[\underline{K}(x_i, x_j)\right]_{i,j=1}^{k}$$

Correlation kernel

Ex: Show that correlation functions characterize a point process.

Show that for any set $A \subseteq \mathbb{Z}$ the following holds:

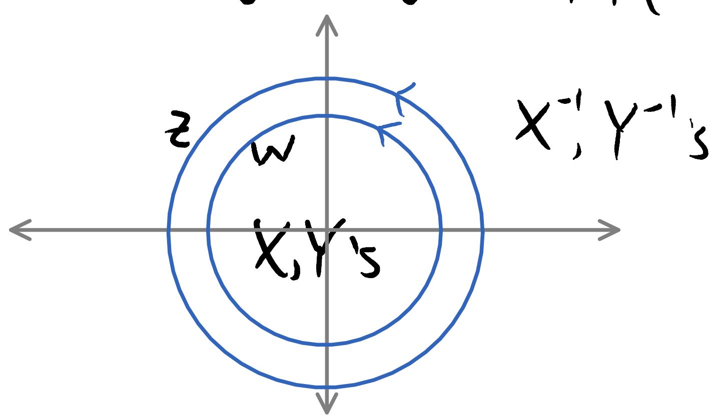
Fredholm det

$$\mathbb{P}(X \cap A = \emptyset) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{x_1, \dots, x_k \in A} \rho_k(x_1, \dots, x_k) =: \det(\underline{I} + \underline{K})_{L^2(A)}$$

Schur measures determinantal kernel

Theorem [Okounkov, 2001] *For* $\lambda=(\lambda_1,\dots,\lambda_N)$ *distributed as* $\mathbb{SM}_{X;Y}$ *the point process* $\{\lambda_1-1,\lambda_2-2,\dots,\lambda_N-N\}$ *is determinantal with kernel*

$$K(i,j)=\frac{1}{4\pi^2}\oint dw\oint dz\,\frac{\Pi(z^{-1};Y)\,\Pi(w;X)}{\Pi(w^{-1};Y)\,\Pi(z;X)}\,\frac{w^i z^{-(j+1)}}{z-w}$$



Other proofs in [Johansson, 2001, Borodin-Rains, 2005].

We will sketch an approach suggested in [Borodin-Corwin, 2011].

Eigenrelations for q-difference operators

For any q, define q-shift operator $(T_{q,x_i} f)(x_1,\dots,x_N) := f(x_1,\dots,qx_i,\dots,x_N)$

Notice that x^m is an eigenfunction of $T_{q,x}$ with eigenvalue q^m.

k^{th} q-difference operator:

multiplication by $V(x)$.

$$D_k = V^{-1} e_k(T_{q,x_1},\dots,T_{q,x_N})\, V$$

$$e_k(u_1,\dots,u_N) = \sum_{i_1<\dots<i_k} u_{i_1}\cdots u_{i_k}$$

Schur polynomials eigenfunctions:

$$(D_k S_\lambda)(x_1,\dots,x_N) = e_k\left(q^{N-1+\lambda_1}, q^{N-2+\lambda_2},\dots, q^{\lambda_N}\right) S_\lambda(x_1,\dots,x_N)$$

Ex: Prove this relation.

Computing expectations

Lets focus on first q-difference operator. It can be written as

$$D_1 = \sum_{i=1}^{N} \prod_{j \neq i}^{N} \frac{q x_i - x_j}{x_i - x_j} T_{q, x_i}$$

Recalling $D_1 s_\lambda(X) = \left(\sum_{j=1}^{N} q^{N-j+\lambda_j}\right) s_\lambda(X)$, the following recipe allows us to compute certain expectations

$$\mathbb{E}\left[\sum_{j=1}^{N} q^{N-j+\lambda_j}\right] = \frac{D_1^{(X)} \Pi(X;Y)}{\Pi(X;Y)}$$

Integral formulas for expectations

We can encode application of first q-difference operator on multiplicative functions $F(u_1,\dots,u_N) = f(u_1)\cdots f(u_N)$ as contour integrals

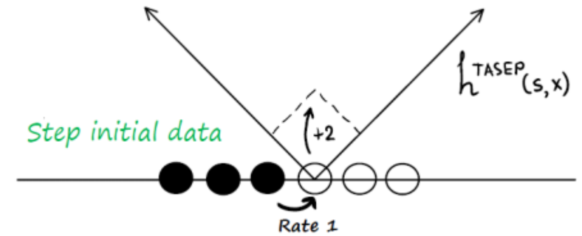
$$\frac{(D_1F)(X)}{F(X)} = \sum_{i=1}^{N} \prod_{j\neq i}^{N} \frac{qx_i - x_j}{x_i - x_j}\,\frac{f(qx_i)}{f(x_i)} = \frac{1}{2\pi i}\oint_{\text{around }\{x_i\}} dw \prod_{j=1}^{N} \frac{qw - x_j}{w - x_j}\,\frac{1}{qw - w}\,\frac{f(qw)}{f(w)}$$

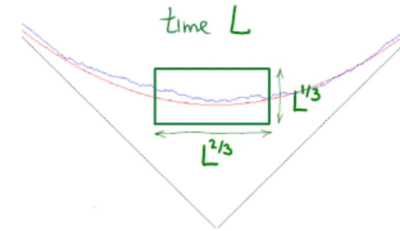
But q was arbitrary. Can extract one-point correlation function

$$\mathbb{P}\left(n \in \{\lambda_j - j\}_{j=1}^{N}\right) = \oint_{|\varepsilon|} \frac{dq}{2\pi i}\, q^{-N-n-1}\, \mathbb{E}\left[\sum_{j=1}^{N} q^{N-j+\lambda_j}\right]$$

Can appeal to higher q-difference operators to prove theorem.

Application: TASEP fluctuations





$$h_L(t,x) := \frac{1}{L^{1/3}}\, h^{TASEP}(Lt, L^{2/3}x) - L^{2/3}\frac{t}{2}$$

Theorem *[Johansson, 1999]*

For TASEP with step initial data

$$\lim_{L\to\infty} \mathbb{P}\{h_L(1,0) \geq -s\} = F_{GUE}(s)$$

Tracy-Widom limit distribution for the largest eigenvalue of large Hermitian matrices

One proof follows by taking steepest descent asymptotics of Fredholm determinant provided by connection to Schur measure. Naturally leads to Fredholm determinant formula for $F_{GUE}(s)$:

$$F_{GUE}(s) = 1 + \sum_{n\geq 1} \frac{1}{n!} \int_s^\infty dx_1 \cdots \int_s^\infty dx_k \det\left[K(x_i,x_j)\right]_{i,j=1}^n, \quad K(x,y) = \int_0^\infty dr\, Ai(x+r)Ai(y+r)$$

Lecture 2 summary

- Schur measure and process generalize GUE corners process
- Diaconis-Fill type dynamics provide link to TASEP (like Warren's)
- Determinantal structure leads to explicit formulas /asymptotics

Lecture 3 preview

- Macdonald measure and process generalizes Schur process
- Structure of Macdonald polynomials leads to integrable particle systems (e.g. q-TASEP, stochastic heat and KPZ equations...)
- Eigenrelations satisfied by Macdonald polynomials leads to explicit formulas for expectations of observables and certain asymptotics